Gravity field determination using post-Newtonian energy integrals

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ORIGINAL ARTICLE

On the energy integral for first post-Newtonian approximation

Joseph O'Leary[1,2] · James M. Hill[1] · James C. Bennett[2,3]

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Abstract The post-Newtonian approximation for general relativity is widely adopted by the geodesy and astronomy communities. It has been successfully exploited for the inclusion of relativistic effects in practically all geodetic applications and techniques such as satellite/lunar laser ranging and very long baseline interferometry. Presently, the levels of accuracy required in geodetic techniques require that reference frames, planetary and satellite orbits and signal propagation be treated within the post-Newtonian regime. For arbitrary scalar W and vector gravitational potentials $W^j (j = 1, 2, 3)$, we present a novel derivation of the energy associated with a test particle in the post-Newtonian regime. The integral so obtained appears not to have been given previously in the literature and is deduced through algebraic manipulation on seeking a Jacobi-like integral associated with the standard post-Newtonian equations of motion. The new integral is independently verified through a variational formulation using the post-Newtonian metric components and is subsequently verified by numerical integration of the post-Newtonian equations of motion.

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ORIGINAL ARTICLE

Post-Newtonian satellite orbits

Joseph O'Leary[1,2] · James M. Hill[1]

Received: 4 August 2018 / Accepted: 11 September 2018

Abstract The first post-Newtonian approximation of general relativity is used to account for the motion of solar system bodies and near-Earth objects which are slow moving and produce weak gravitational fields. The n-body relativistic equations of motion are given by the Einstein-Infeld-Hoffmann equations. For $n = 2$, we investigate the associated dynamics of two-body systems in the first post-Newtonian approximation. By direct integration of the associated planar equations of motion, we deduce a new expression that characterises the orbit of test particles in the first post-Newtonian regime generalising the well-known Binet equation for Newtonian mechanics. The expression so obtained does not appear to have been given in the literature and is consistent with classical orbiting theory in the Newtonian limit. Further, the accuracy of the post-Newtonian Binet equation is numerically verified by comparing secular variations of known expression with the full general relativistic orbit equation.

Weinberg 1972). The field equations of GR are a sy
ten, non-linear, coupled partial differential equations
obtaining exact analytical solutions (Stephani et al.
a notoriously formidable task. Notable solutions ar
by Schwarzschild (1916) and Kerr (1963) which
highly ideal gravitational systems for non-rotating,
cally symmetric and rotating, axially symmetric blac
respectfully.

As early as 1916, Einstein hypothesised (Einstei
1918) the existence of gravitational waves using a
field approximation (d'Inverno 1992; Misner et al
Weinberg 1972) of the field equations of gravity. T
direct observation of gravitational waves was due to
alescence of a binary black hole system (Abbott et al
which was detected by the advanced laser interfe
gravitational-wave observatory collaboration (Abbo
2009; Harry 2010). More recently, a simultaneous c
tion of both gravitational waves (Abbott et al. 20

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Some background, relevant results & motivation

Academic background

- Applied mathematician (Dublin, Ireland).
- Completed honours year in General Relativity.
- Began PhD (University of South Australia) July 2016 and expect to finish early – mid July 2019.
- Member of Research program 3 for Space Environment Research Centre, Canberra.
- Primary interests in relativistic mechanics in the post-Newtonian approximation for general relativity.

Some published results relevant for talk

Energy Integral (O'Leary, Hill & Bennett, 2018)

$$v^2 = \frac{5c^2}{9}\left(1 - e^{-6U/c^2}\right) - \frac{4U}{3} + 2\mathcal{E}e^{-6U/c^2}$$

Post-Newtonian orbit equation (O'Leary & Hill, 2018)

$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{16\mu^2}{3h_0^2c^2}e^{8\mu u/c^2}\right)u = \frac{14\mu}{9h_0^2}e^{8\mu u/c^2} - \frac{5\mu}{9h_0^2}e^{2\mu u/c^2}$$

Some background, relevant results & motivation

$$v^2 = \frac{5c^2}{9}\left(1 - e^{-6U/c^2}\right) - \frac{4U}{3} + 2\mathcal{E}e^{-6U/c^2}$$

Using $U/c^2 \ll 1$, the exponential functions can be expanded using Taylors series approximation and produce known and well established results, namely

$$v^2 = 2U + 2\mathcal{E}(1 - 6U/c^2) - 10U^2/c^2$$

Both results can be seen as a formal extension of the classical Newtonian conservation of energy to include PN contributions.

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Some background, relevant results & motivation

Post- Newtonian Binet equation (O'Leary & Hill, 2018)

$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{16\mu^2}{3h_0^2c^2}e^{8\mu u/c^2}\right)u = \frac{14\mu}{9h_0^2}e^{8\mu u/c^2} - \frac{5\mu}{9h_0^2}e^{2\mu u/c^2}$$

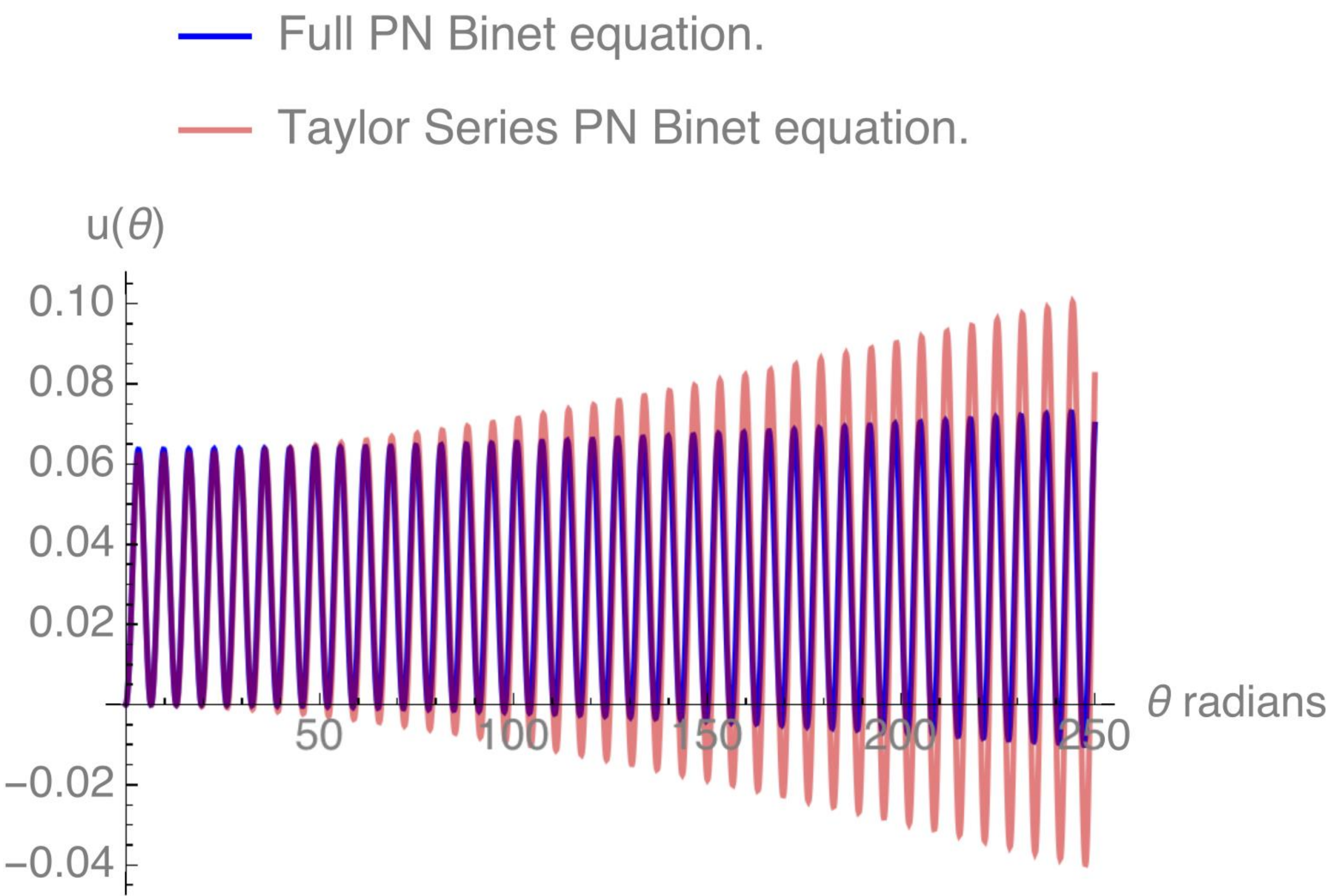
Taylor series →

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{6\mu^2}{h_0^2c^2}\right)u = \frac{\mu}{h_0^2}$$

Which orbit equation more closely resembles full non-linear GR?

Non-linear orbit equation GR

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h_0^2} + \frac{3\mu}{c^2}u^2$$



Gravity Field determination

Taken some inspiration from

- **On the principles of satellite-based Gravity Field Determination with special focus on the Satellite Laser Ranging technique (**Förste , König , Bruinsma, Lemoine , Dahle , Reinquin and Flechtner) (IWLR 2016)
- **Gravity Field Analysis from the Satellite Missions CHAMP and GOCE** (Martin K. Wermuth) (PhD thesis, Institut fur Astronomische und Physikalische Geodasie)
- **Gravity: Newtonian, post-Newtonian, Relativistic** (Poisson & Will, 2014)
- **Satellite Orbits: Models, Methods, Applications** (Montenbruck, Gill, 2012)
- **International Centre for Global Earth Models (ICGEM)** (Barthelmes & Köhler, 2016)
- **Definition of the functionals of the geopotential and their calculations from spherical harmonic models** (Barthelmes, 2013)
- **CHAMP gravity field recovery with energy balance approach: First results** (Gerlach, Sneeuw, Visser, Svehla, 2003)
- **An application of Jacobi's integral to the motion of an earth satellite** (O'Keefe, 1957)

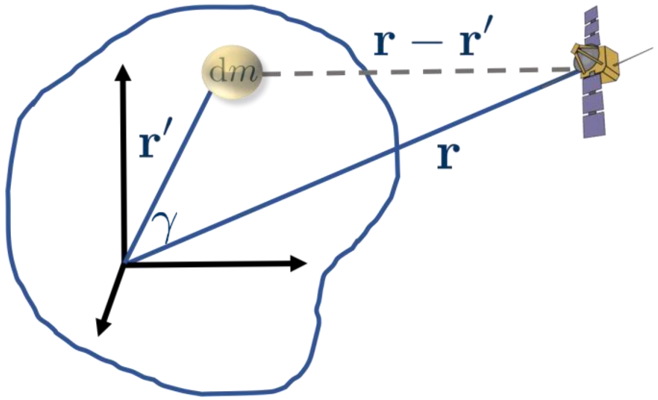
Gravity Field determination – Potential theory

Two methods used for decomposition of the Earth gravitational potential given by:

- Spherical harmonic expansion.
- Symmetric trace-free tensors (not as common in classical literature. However, useful mathematical tool).

Spherical Harmonic decomposition

Newtonian gravitational potential for arbitrary mass distribution



$$U = G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

Legendre polynomial generating function given by

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \left[r^2 - 2\left(\mathbf{r} \cdot \mathbf{r}'\right) + \left(r'\right)^2\right]^{-1/2} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l \mathrm{P}_l\left(\cos\gamma\right)$$

Introducing spherical polar coordinates and making use of the addition and Rodrigues formulae. We express the potential in well known form according to

$$U = \frac{GM}{r} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{a}{r}\right)^l P_{lm}\left(\sin\phi\right)\left(C_{lm} \cos\left(m\lambda\right) + S_{lm} \sin\left(m\lambda\right)\right)$$

Gravity Field determination – Potential theory

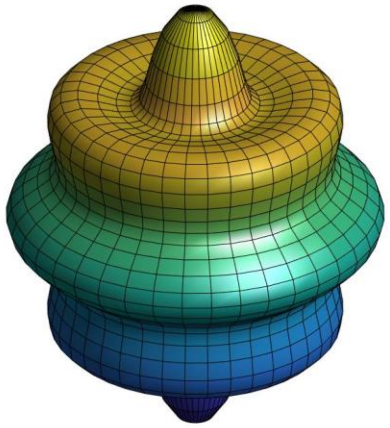
$$U = \frac{GM}{r} \sum_{l=0}^{N} \sum_{m=0}^{l} \left(\frac{a}{r}\right)^{l} P_{lm}\left(\sin\phi\right)\left(C_{lm}\cos\left(m\lambda\right) + S_{lm}\sin\left(m\lambda\right)\right)$$

Represents an Earth gravity field model of maximum degree N. The accuracy depends on the determination of the spherical harmonic coefficients highlighted in red.

Examples of different kinds of spherical harmonics

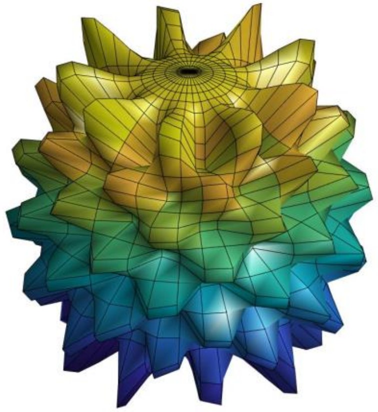
Zonal

$l = 8,\ m = 0$



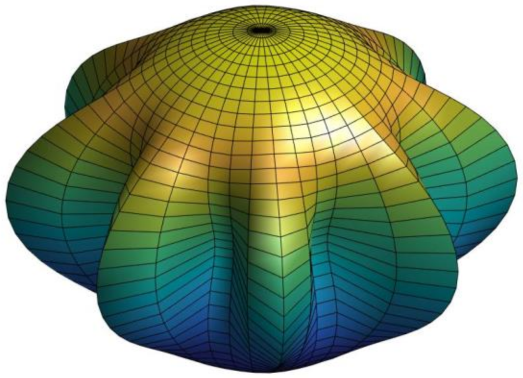
Tesseral

$l = 16,\ m = 9$



Sectorial

$l = 8,\ m = 8$



Gravity Field determination – Potential theory

The International Centre for Global Earth Models (ICGEM) (http://icgem.gfz-potsdam.de/home)

- Collects and archives global gravity field models
- Provides reports introducing ideas such as spherical harmonic expansions and defining geopotential functionals.
- Web-based interactive visualisations for various gravity field models

Nr	Model	Year	Degree	Data	References	Download	Calculate	Show	DOI
168	Tongji-Grace02k	2018	180	S(GRACE)	Chen, Q. et al, 2018	gfc zip	Calculate	Show	✔
167	SGG-UGM-1	2018	2159	EGM2008, S(GOCE)	Liang, W. et al., 2018 & Xu, X. et al. (2017)	gfc zip	Calculate	Show	✔
166	GOSG01S	2018	220	S(GOCE)	Xu, X. et al., 2018	gfc zip	Calculate	Show	✔
165	IGGT_R1	2017	240	S(GOCE)	Lu, B. et al, 2017	gfc zip	Calculate	Show	✔
164	IfE_GOCE05s	2017	250	S(GOCE)	Wu, H. et al, 2017	gfc zip	Calculate	Show	✔
163	GO_CONS_GCF_2_SPW_R5	2017	330	S(GOCE)	Gatti, A. et al, 2016	gfc zip	Calculate	Show	✔
162	GAO2012	2012	360	A, G, S(GOCE), S(GRACE)	Demianov, G. et al, 2012	gfc zip	Calculate	Show	✔
161	XGM2016	2017	719	A, G, S(GOCO05s)	Pail, R. et al, 2017	gfc zip	Calculate	Show	✔
160	Tongji-Grace02s	2017	180	S(Grace)	Chen, Q. et al, 2016	gfc zip	Calculate	Show	✔
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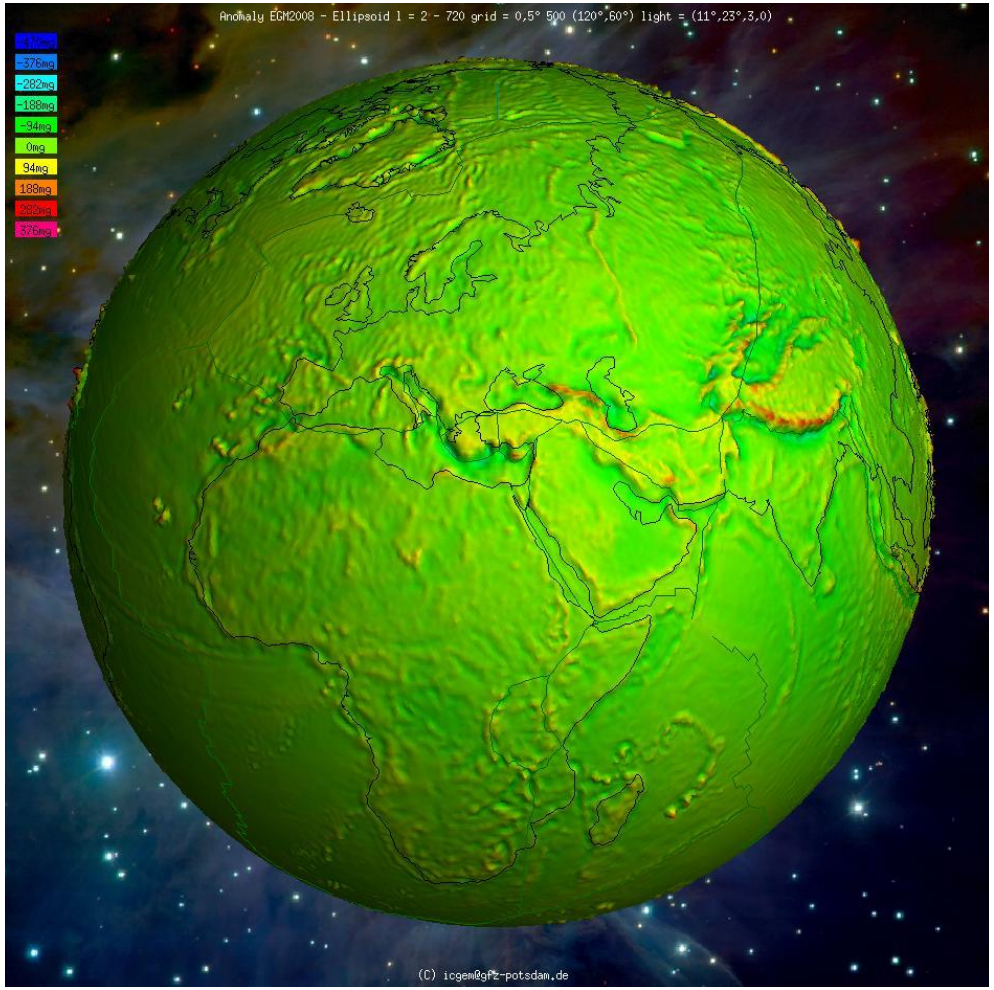
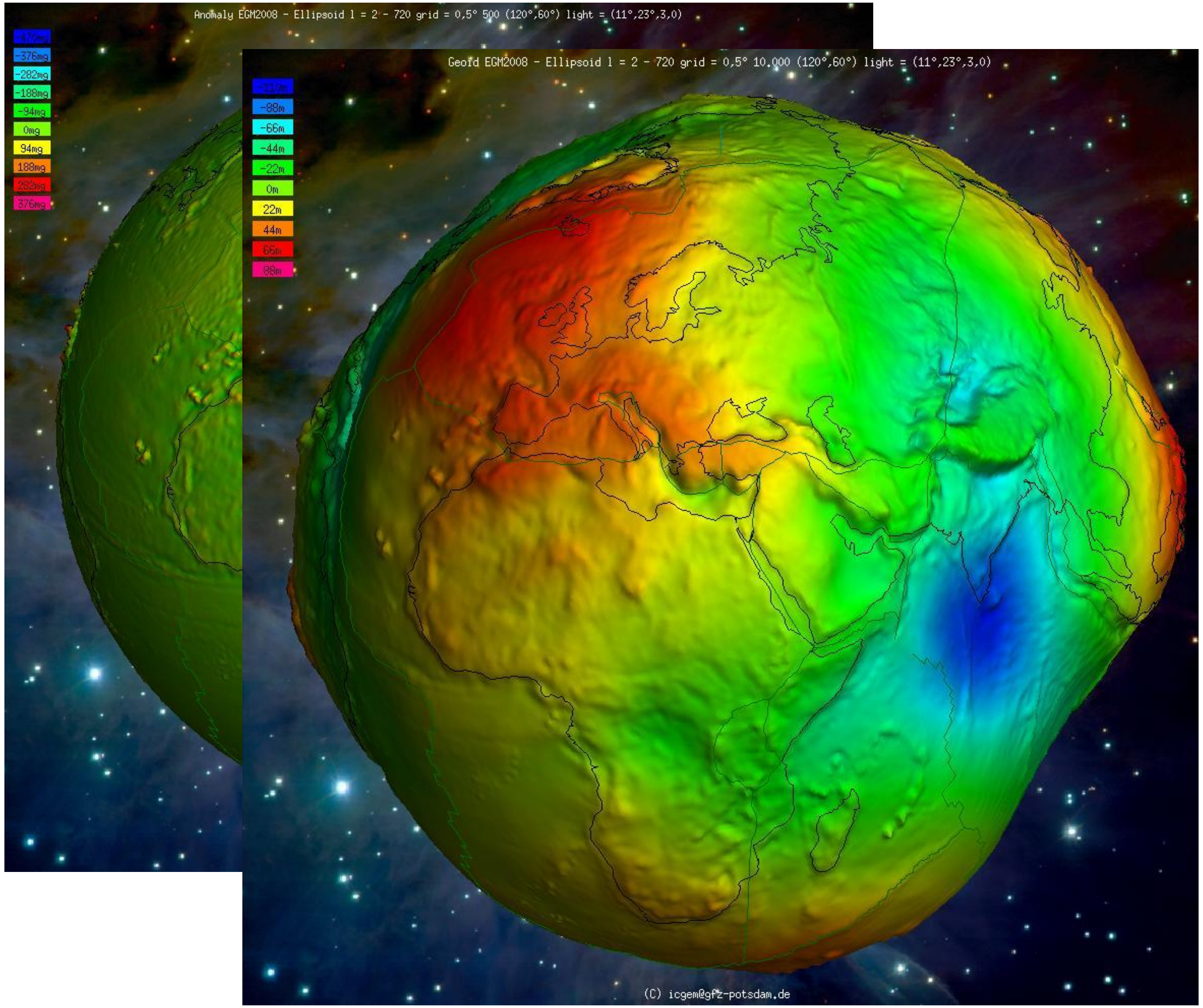
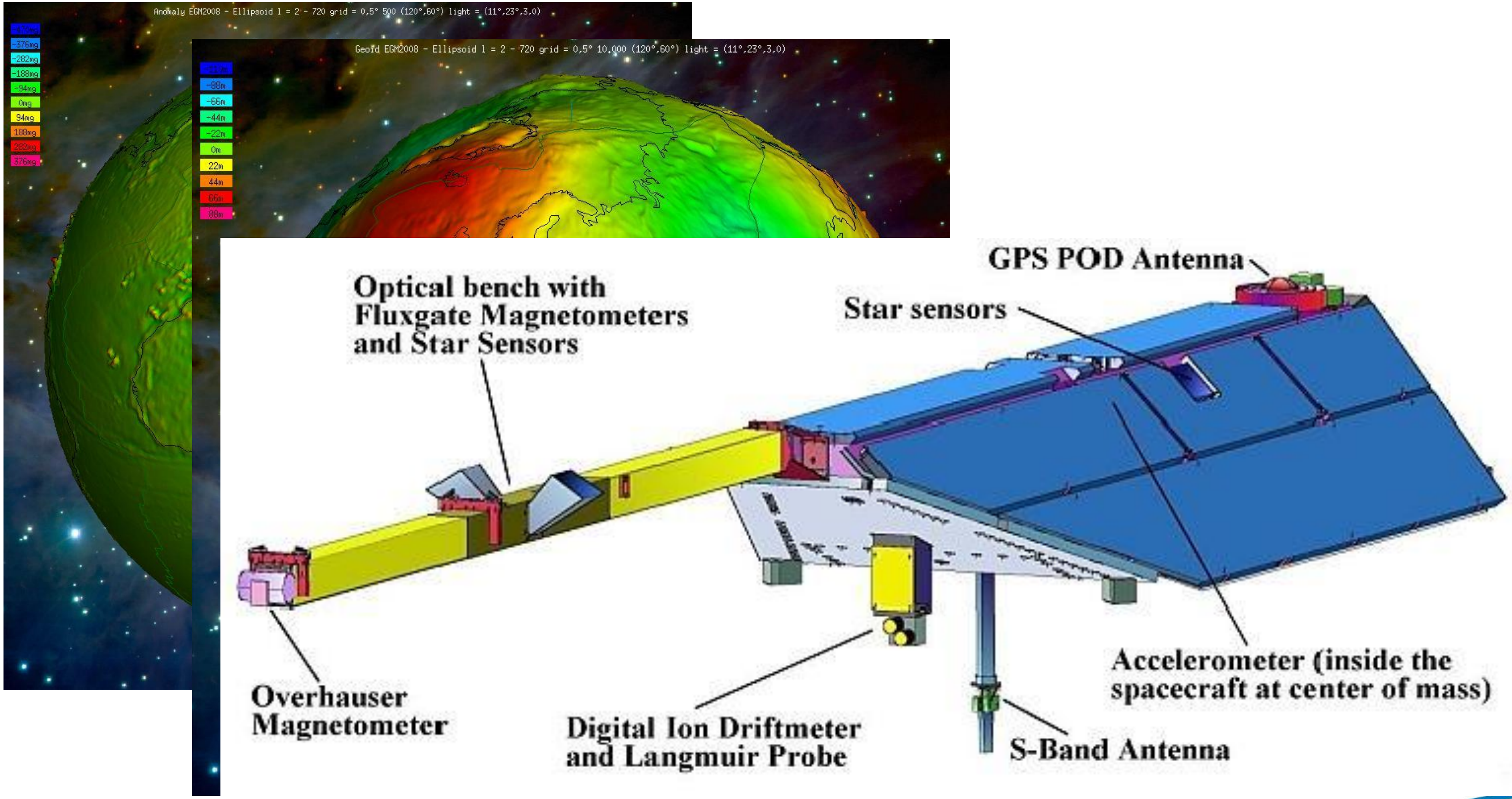
Gravity Field determination – Potential theory



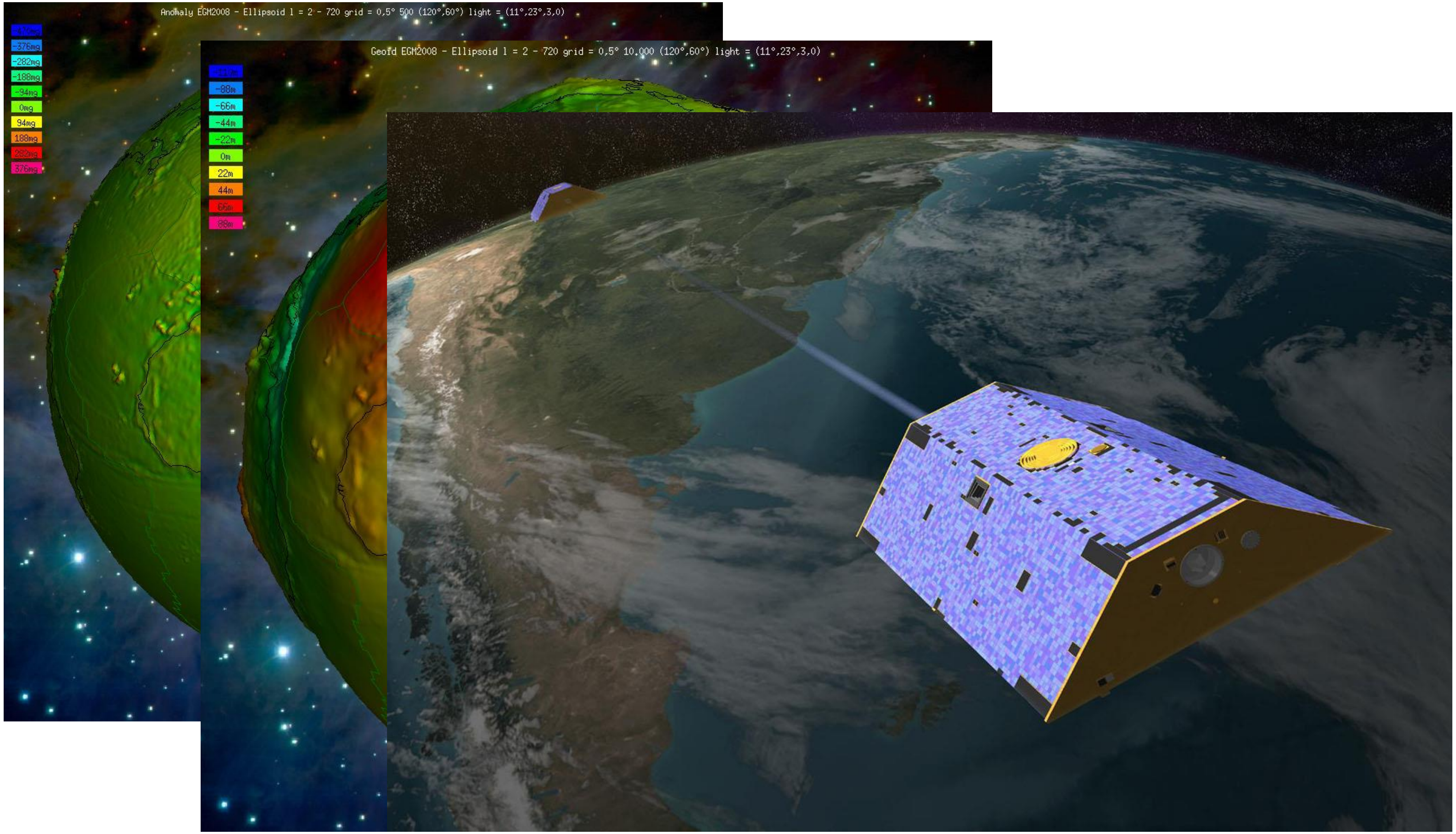
Gravity Field determination – Potential theory



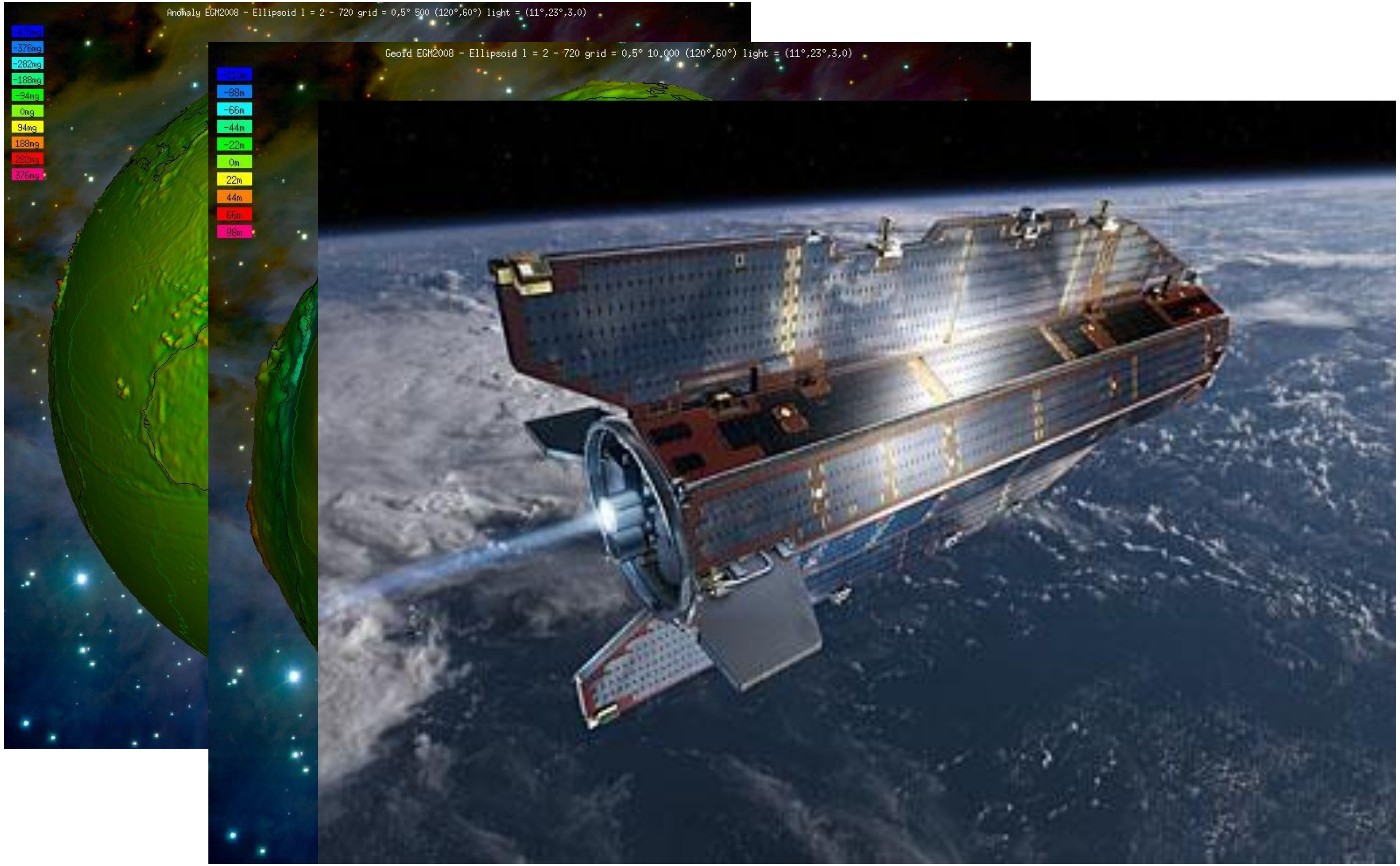
Gravity Field determination – Potential theory



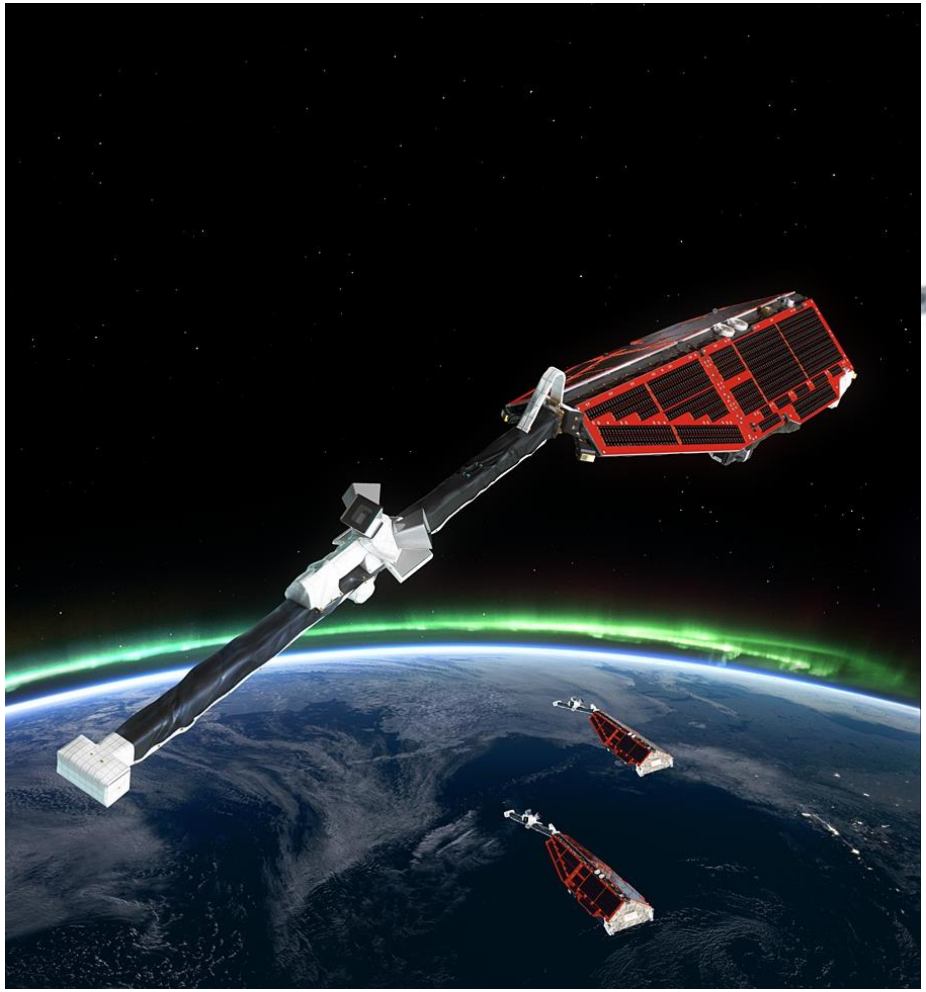
Gravity Field determination – Potential theory



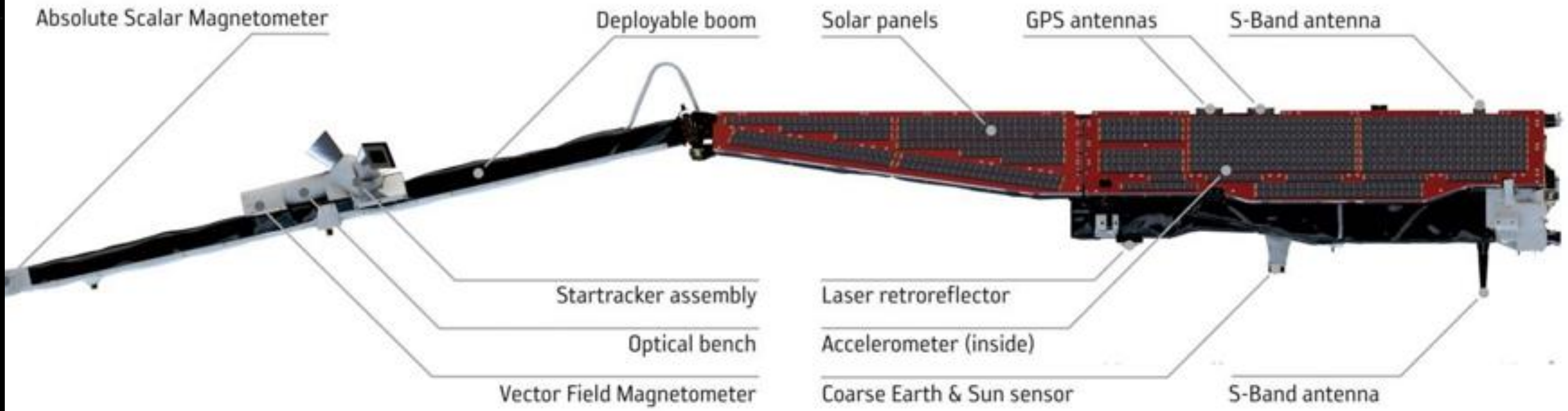
Gravity Field determination – Potential theory



ESA Swarm mission for gravity field determination



(image copyright: ESA)



(image copyright: ESA)

- Swarm missions consists of three identical satellites (A, B and C) launched November 2013.
- On – board sensors include (but not limited to) an accelerometer, GPS receiver and laser retro reflector.
- In the interim of missions such as GRACE, GOCE and CHAMP. The swarm constellation provided an ample opportunity for the geodetic community for continued gravity field determination with a wealth of literature already dedicated to such research.

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Energy balance approach for gravity field determination

The conservation of energy arises as a first integral of Newton's second law of motion namely,

$$\frac{d^2x^i}{dt^2} = \frac{\partial U}{\partial x^i}$$

where following multiplication by the velocity vector

$$\frac{dx^i}{dt}\frac{d^2x^i}{dt^2} - \frac{dx^i}{dt}\frac{\partial U}{\partial x^i} = 0$$

is immediately integrable to give

$$\frac{1}{2}v^2 - U = C$$

Constant of the motion for a satellite in an inertial (non-accelerating) reference frame subject to a conservative field (work done is independent of path taken).

$$U_D = \frac{1}{2}v^2 - U_R - \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{r})^2 - \int \mathbf{a}_s \cdot d\mathbf{r} - \int \mathbf{a}_t \cdot d\mathbf{r} - C$$

Energy balance approach for gravity field determination

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Energy balance approach for gravity field determination

$$U_D = \frac{1}{2}v^2 - U_R - \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{r})^2 - \int \mathbf{a}_s \cdot d\mathbf{r} - \int \mathbf{a}_t \cdot d\mathbf{r} - C$$

Disturbing potential signifying perturbations/disturbances from a given reference ellipsoid

Potential for some Reference ellipsoid

Centrifugal potential

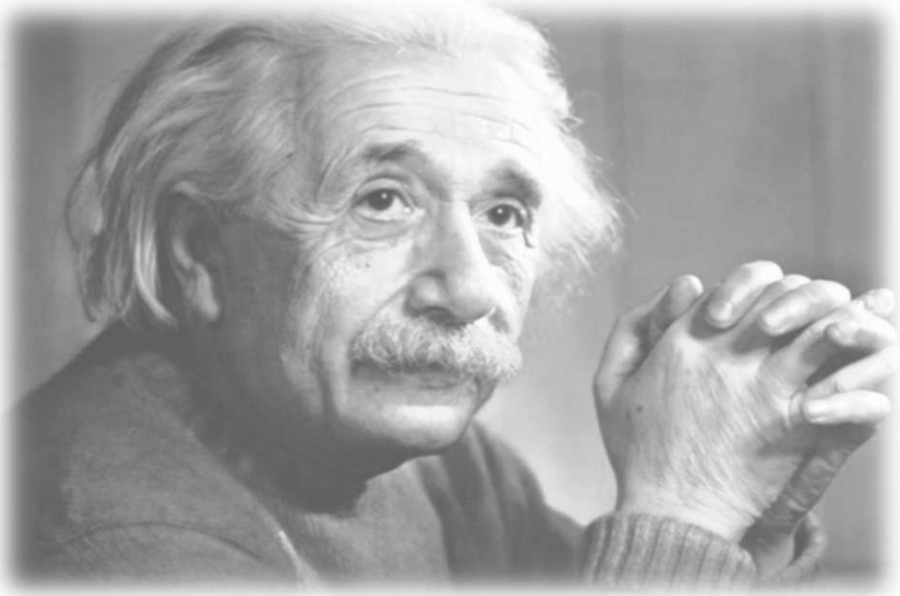
Surface and time varying accelerations (non-conservative) which are measured by on-board accelerometers

It is important to note that the formal transformation from an inertial to a non-inertial or Earth-fixed frame gives rise to both Coriolis and Euler accelerations

$$\int (\boldsymbol{\omega} \times \dot{\mathbf{r}}) \cdot \dot{\mathbf{r}}\, \mathrm{d}t \qquad \int (\dot{\boldsymbol{\omega}} \times \mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

respectively. The time-variability of the Earth's rotation is negligible and can be neglected and the velocity vector is perpendicular to the Coriolis acceleration.

Post-Newtonian energy integral



The field equations of GR* are notoriously difficult to solve

- Tensorial form.
- System of 10 non-linear, coupled partial differential equations.
- Very few solutions can be used to model physically realisable situations.
- Solutions that do exist exhibit very high degrees of symmetry.

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$
$$G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} R/2.$$

The Post-Newtonian approximation

The gravitational field of the Earth is weak when compared with some of general relativity's more exotic phenomena i.e. we are not describing gravitational physics at the event horizon of a black hole.

Objects move slow when compared with the speed of light e.g. orbital velocity of a GPS satellite is approximately $4\ km/s$.

The metric tensor components can be thought of as a small perturbation from the flat Minkowski space of special relativity.

A linearised form of the field equations called the Post-Newtonian (PN) approx. was developed for the description of solar system dynamics and is coined “unreasonably effective”.

Post-Newtonian energy integral

PN metric tensor components

$$g_{00} = -1 + 2\frac{U}{c^2} - 2\frac{U^2}{c^4},$$
$$g_{0j} = -\frac{4U^j}{c^3},$$
$$g_{jk} = \delta_{jk}\left(1 + 2\frac{U}{c^2}\right).$$

Christoffel symbols of the second kind

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{g^{\alpha\sigma}}{2}\left(\partial_\gamma g_{\sigma\beta} + \partial_\beta g_{\sigma\gamma} - \partial_\sigma g_{\beta\gamma}\right)$$

“modified” geodesic equation of motion

$$\frac{d^2 x^\alpha}{dt^2} = -\left(\Gamma^{\alpha}_{\beta\gamma} - \frac{v^\alpha}{c}\Gamma^{0}_{\beta\gamma}\right) v^\beta v^\gamma$$

PN equations of motion

$$\frac{d^2 x^j}{dt^2} = \partial_j U + \frac{1}{c^2}\left[\left(v^2 - 4U\right)\partial_j U - 4v^j v^k \partial_k U - 4(\partial_j U^k - \partial_k U^j)v^k\right].$$

Newtonian contribution — Schwarzschild contribution — Lense-Thirring contribution

Post-Newtonian energy integral

By following a formally identical procedure we may derive the energy integral associated with the PN approximation for GR. The equations of motion for a near-Earth object in the PN regime are given by

$$\frac{d^2x^j}{dt^2} = \partial_j U + \frac{1}{c^2}\left[\left(v^2 - 4U\right)\partial_j U - 4v^j v^k \partial_k U - 4(\partial_j U^k - \partial_k U^j)v^k\right].$$

Multiplying the above expression by v^j yields

$$\frac{d}{dt}\left[\frac{v^2}{2} - U + \frac{2U^2}{c^2}\right] = \frac{1}{c^2}\left[-3v^2\frac{dU}{dt} - 4v^k\frac{dU^k}{dt} + 4\frac{dU^j}{dt}v^j\right]$$

where it is clear that the final two terms cancel to give

$$\frac{d}{dt}\left[\frac{v^2}{2} - U + \frac{2U^2}{c^2}\right] = -\frac{3v^2}{c^2}\frac{dU}{dt}$$

which can be solved analytically to give

$$\boxed{v^2 = \frac{5c^2}{9}\left(1 - e^{-6U/c^2}\right) - \frac{4U}{3} + 2\mathcal{E}e^{-6U/c^2}}$$

Post-Newtonian energy integral

$$v^2 = \frac{5c^2}{9}\left(1 - e^{-6U/c^2}\right) - \frac{4U}{3} + 2\mathcal{E}e^{-6U/c^2}$$

- The energy constant is identified in the Newtonian limit by demanding we recover the classical law of conservation of energy.
- Expanding the exponentials to PN order gives

$$v^2 = 2U + 2\mathcal{E}(1 - 6U/c^2) - 10U^2/c^2$$

- In a completely analogous way to that of the Newtonian energy integral we are proposing using this form of energy conservation for gravity field recovery.
- This is to be done in conjunction with precise orbit information from on-board GPS receivers and accelerometers for ESA's swarm mission.

Potential problem areas and complications before proceeding with analysis

- The derivation of post-Newtonian first integrals contain some ambiguities. Popular literature (Poisson & Will, 2014) states that these quantities are actually not unique and are provided modulo an arbitrary amount of

$$c^{-2}\left(v^2 - 2U\right)^2$$

Which is constant by virtue of Newtonian mechanics. A manuscript (O'Leary & Hill 2018b) is currently in preparation which discusses this issue and what effect it may have on overall dynamics.

- The description of arbitrarily shaped bodies in the PN formalism is given by the so called Blanchet Damour moments rather than the usual classical spherical harmonic decomposition. Again, this requires further elucidation before proceeding.

Conclusion

The energy integral method outlined for gravity field recovery is over half a century old (O'Keefe, 1957). However, such methods are only made possible with data provided by dedicated gravity field missions in recent years. Relativity at the PN level is abundant in many areas of navigation, geodesy, definition of reference frames, interplanetary missions and so on. Here we presented a potential method of generalising well-known gravity field determination methods to include PN contributions.