# Optimizing the Observation Schedule in Laser Ranging Experiments

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## Abstract

The thirty years of accumulated lunar laser ranging (LLR) data is substantially non-uniform in its distribution with respect to lunar synodic phase. As a consequence, new LLR observations show extreme variation in their contribution toward further improvements in the estimation precision of certain scientifically interesting parameters. This can be quantified by construction of a *worth function* for model parameters. Specific cases in LLR and SLR are discussed as examples — the Equivalence Principle violating parameter of gravitational theory measured in LLR, and the ranging site altitude parameters measured in SLR. More optimum observation schedules for achieving various science products from laser ranging missions can emerge from construction and examination of the worth functions for model parameters of interest.

## 1 Introduction

In programs such as laser ranging to near-Earth satellites (SLR), to the Moon (LLR), or in future ranging at the interplanetary scale, the ranging data must generally be fit to a complex model containing a large number of fit-for parameters. But the science products of these ranging programs will generally consist of the estimated values of only a few of the many parameters. Most parameters are simply a necessary part of a complete model for the range measurements; their particular value being of little significance. If there is a cost (money, opportunity, resource committments, etc.) incurred in making ranging measurements, then it is important to know the relative values, the *worths* if you will, of different observations for achieving the science goals, so that more optimum observation schedules can be suggested and implemented.

Suppose one wishes to estimate some parameter $P$ using a traditional least-squares fit procedure, with this parameter making a contribution to the measured range $R(t)$

$$\delta R(t) = \delta P \; f(t) \; + \; ... \tag{1}$$

with ... indicating contributions from other model parameters. If this parameter's *partial function* $f(t)$ would happen to be orthogonal to all the other parameter partial functions, then the well known common sense rule is that one makes observations at times when the partial function for the parameter of interest has its largest absolute value, because the improvement in knowledge of the parameter is in this unusual case

$$\delta < (\delta P)^2 > \sim \; - \; | \; f(t)^2 \; | \; / \sigma^2 \tag{2}$$

with $\sigma$ being the root-mean-square error associated with the new observation.

But if the partial functions for the other model parameters are not orthogonal to that of the particular parameter of interest, the situation is somewhat different. In this more realistic situation, I have recently shown that only that part of the partial function $f(t)$ which is orthogonal to all other model partials is operable for improving knowledge of the parameter $P$ [8]. This effective part of the parameter's partial function is

$$f(t)^* \;\equiv\; f(t) - \sum_{m=1}^{M-1} \vec{f}(t) \cdot \hat{u}_m \, \hat{u}_m \tag{3}$$

with $\hat{u}_m$ being a set of function unit vectors which spans the space of partial functions for the other $M-1$ parameters of the ranging model. The indicated scalar products are given by

$$\vec{a} \cdot \vec{b} \;\equiv\; \sum_{i=1}^{N} a(t_i) \, b(t_i) / \sigma_i^2 \tag{4}$$

with there being $N$ total range measurements made at the specific times $t_i$.

## 2 Lunar Laser Ranging

LLR provides a dramatic example of how unexpectedly variable can be the relative values of range observations. Seeking to enhance this ungoing thirty year program was, in fact, my motivation for investigating this issue. The very non-uniform distribution of past LLR observations is shown in Figure 1. when plotted as a function of the Moon's synodic phase. Because of several technical difficulties in making observations near new and full moon phase, the observation density is peaked near quarter moon phases. But on examining this distribution more closely, it is further seen that a significant asymmetry of the data density about quarter moon phase exists. This has detrimental consequences for the science product, and there seems to be no strong technical necessity for this latter situation. A fourier representation of the distribution shown in Figure 1. is

$$n(D) \;=\; n_o \left( 1 \;+\; \sum_{n=1}^{\infty} \left( C_n \; cos(nD) \;+\; S_n \; sin(nD) \right) \right) \tag{5}$$

with the found coefficients $C_1 \simeq -.50$, $C_2 \simeq -1.09$, $C_3 \simeq .61$, ..., and $S_1 \simeq .01$, $S_2 \simeq -.20$, $S_3 \simeq .25, ...$ As a result, various model partial functions which would normally be expected to be essentially orthogonal, become non-orthogonal.

LLR data provides a sensitive probe for the question — do different bodies fall at the same rate in a gravitational field [1] [2] [3]? If Earth and Moon have different acceleration rates toward the Sun, then there

Synodic Phase Distribution of LLR Observations

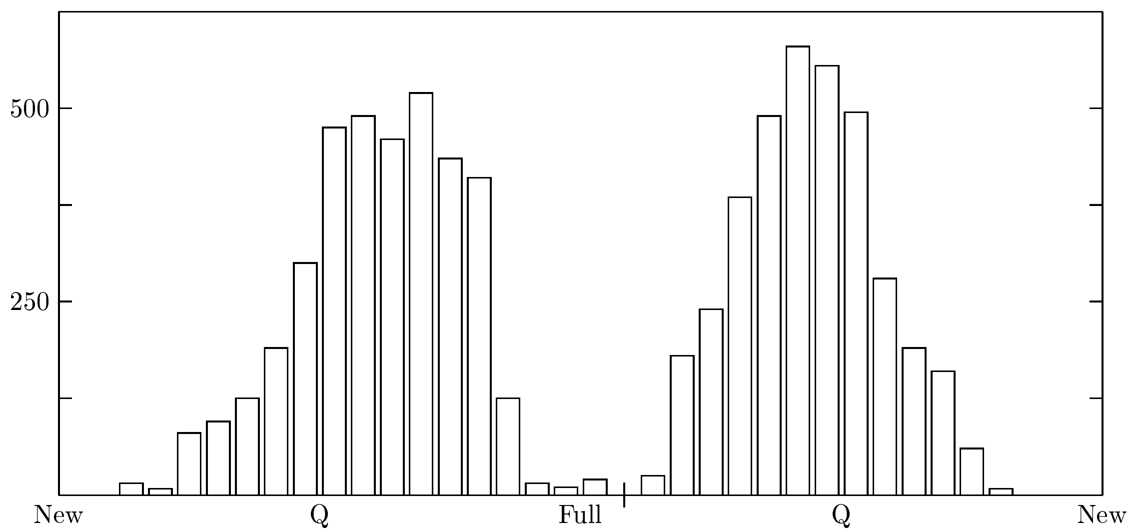


Figure 1: The distribution of 7364 LLR observations made between 1985 and 1997, plotted with respect to synodic phase in $10^o$ bins, shows scarcity near new and full moons and asymmetry about quarter moons.

is an Earth-Moon range contribution which is mainly a synodic month oscillation [6] [7].

$$\delta R(t) \ = \ A \ cosD \ + \qquad \text{small harmonic and eccentric sidebands} \tag{6}$$

with

$$A \ = \ \delta_{em} \ 2.9 \ 10^{12} \ cm \tag{7}$$

and $\delta_{em}$ being the fractional difference in the gravitational acceleration rate of Earth and Moon toward the Sun

$$\delta_{em} \ = \ \frac{|\vec{a}_e - \vec{a}_m|}{|\vec{g}_s|} \tag{8}$$

If this *equivalence principle-violating* (EPV) signal $cosD$ were orthogonal to all the other partial signals of the LLR model, then, as mentioned before, $(cosD)^2$ would give the relative values of observations for measuring the amplitude $A$. This indicates preference for range measurements near new and full moons. Taking into account the technical difficulties previously mentioned, the strategy might then simply be to push the observations toward both new and full moon phases as much as technically possible (without, for example, incurring excessive growth in the measurement errors $\sigma_i$). But there are two other synodically periodic partial signals in the model — 1 and $cos(2D)$ — the first associated with the parameter *mean Earth-reflector distance* and the second associated with size of the solar tide-induced lunar orbit perturbation named the *variation* by Newton who first calculated it over three centuries ago. If there were a uniform density of LLR data, then $cosD$ would be orthogonal to these two signals. But because of the data distribution shown in Figure 1., this is decisively not the case. From Eq. (3) and using the Fourier coefficients found from the data distribution one now finds the orthogonalized EPV signal to be

$$f(t)^*_{EPV} \ \simeq \ cosD \ + \ .54 \ + .53 \ cos(2D) \tag{9}$$

Present Worth Function for LLR's $\cos D$ Amplitude

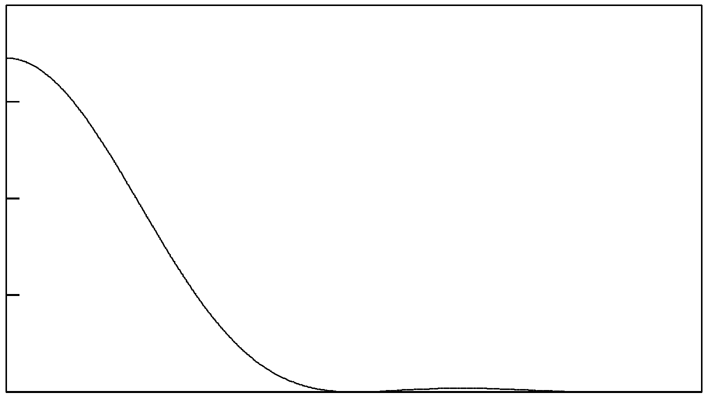


Figure 2: The reduction of formal measurement noise error for the $\cos D$ amplitude from an additional present-day observation made at synodic phase $D$ is shown (in arbitrary units); it strongly suggests that such observations should be made on the new moon side of quarter moon phase.

The square of this orthogonalized function now gives the relative value of a future observation made at synodic phase $D$, given the distribution of past observations. It is plotted in Figure 2 and **very strongly suggests that future observations should be made on the new moon side of quarter moon; additional observations on the full moon side of quarter moon are presently almost worthless in improving the estimation precision of the $EPV$ amplitude!** Examination of this particular orthogonalization process shows that it is not the deficit of observations at new and full moon which creates this situation; it is the past asymmetry of the observation density about quarter moon which produces the strong preference for future observations to be made on the new moon side of quarter moon phase. At some time in the future when the data distribution reestablishes symmetry in this respect, the relative value of observations will then return toward its simple $(cosD)^2$ form.

For testing relativistic theories of gravity, there are two other LLR measurements of particular interest — a possible cosmically varying strength for Newton's coupling parameter $G \rightarrow G(t) \simeq G + \dot{G}(t - t_o)$, and the rate of *geodetic* precession of the local inertial frame which accompanies the Earth-Moon system in its motion through the Sun's gravitational field [4] [5]. The relative value or *worth functions* for these scientific goals have also been investigated elsewhere [8], and strong dependence on the synodic phase is found, with this synodic dependence varying in form from month to month.

So what can we seek in the end? One can imagine the LLR observers consulting a chart of a total *worth function* formed by consolidating those of the several science parameters of interest; this function looking several months into the future and permitting a more optimal scheduling of the observational resources for the purpose of scientific achievement. A doubling, tripling or more of the rate of improvement in estimation precision of selected scientific parameters seems within reach by implementing reasonable adjustments in the schedule of future LLR observations.

## 3 Satellite Laser Ranging

As an introductory application of the *worth function* construction to SLR, consider the problem of optimizing the observation schedule for determination of the terrestrial coordinates for any particular SLR site. Assuming there are many satellites, ranged from many SLR sites, contributing to the total SLR data set, the model for this situation includes four parameters associated with any particular SLR site — its three position coordinates plus a system timing bias. These four parameter partial signals for any site will be substantially non-orthogonal to each other; so I express the total model in the form

$$\delta R_p(t) \;=\; \sum_{n=1}^{4} \delta P_n \; f_n(p,t) \;+ ... \tag{10}$$

$R_p(t)$ is range from a specific SLR station, $p$ indicates a specific satellite passage characterized by several attributes such as satellite altitude and orbital inclination, distance of closest approach of the passage ground track, etc.; and ... indicates all the many other model parameters and their partial signals which will be essentially orthogonal to the four named partial signals and therefore ignorable for purposes here.

Consider, for example, the SLR site's vertical coordinate $\delta z = \delta P_1$; then the orthogonalized partial signal for this coordinate will be

$$f_1(p,t)^* = f_1(p,t) - \sum_{n=2}^{4} c_n \; f_n(p,t) \tag{11}$$

for some determinable choice of the three coefficients $c_n$. The square of this orthogonalized partial signal gives the worth function $W_z(t)$ for range observations during a specific satellite passage as a function of time along that passage, and under some reasonable simplifications turns out to be

$$W_z(t) \;\simeq\; (h/D(t) - < h/D >)^2 \, / \sigma(t)^2 \tag{12}$$

with $h$, $d$, $v$ and $\sigma(t)$ being the particular satellite altitude, distance of closest approach of ground track, speed, and range measurement error, respectively, during the satellite passage by the site, and

$$D(t) \;=\; \sqrt{h^2 + d^2 + v^2 t^2} \tag{13}$$

in which $< h/D >$ is that quantity's average value over all previous range measurements from the site to all satellites. More precise expressions for the worth functions of a variety of scientifically interesting parameters obtainable from SLR are under development.

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