Relativistic Aspects of SLR/LLR Geodesy

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Content:

- Introduction to relativity
- Solving Einstein's equations
- Gauge freedom
- Global and local coordinates
- Gauge freedom of EIH equations
- Toward better SLR/LLR relativistic modelling
- Relativistic geoid

Relativity for a Layman

Put your hand on a hot stove for a minute, and it seems like an hour.

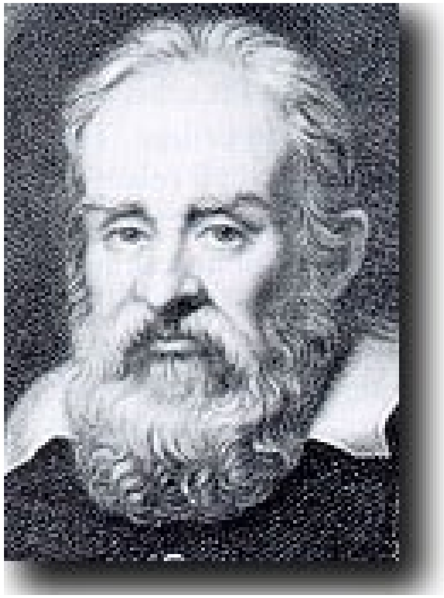
Sit with a pretty girl for an hour, and it seems like a minute. That's relativity!

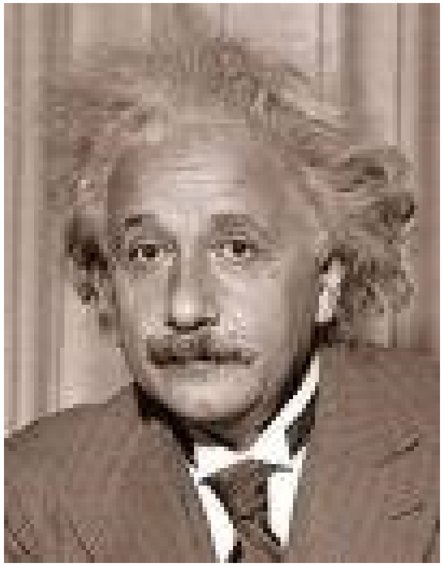
A. Einstein

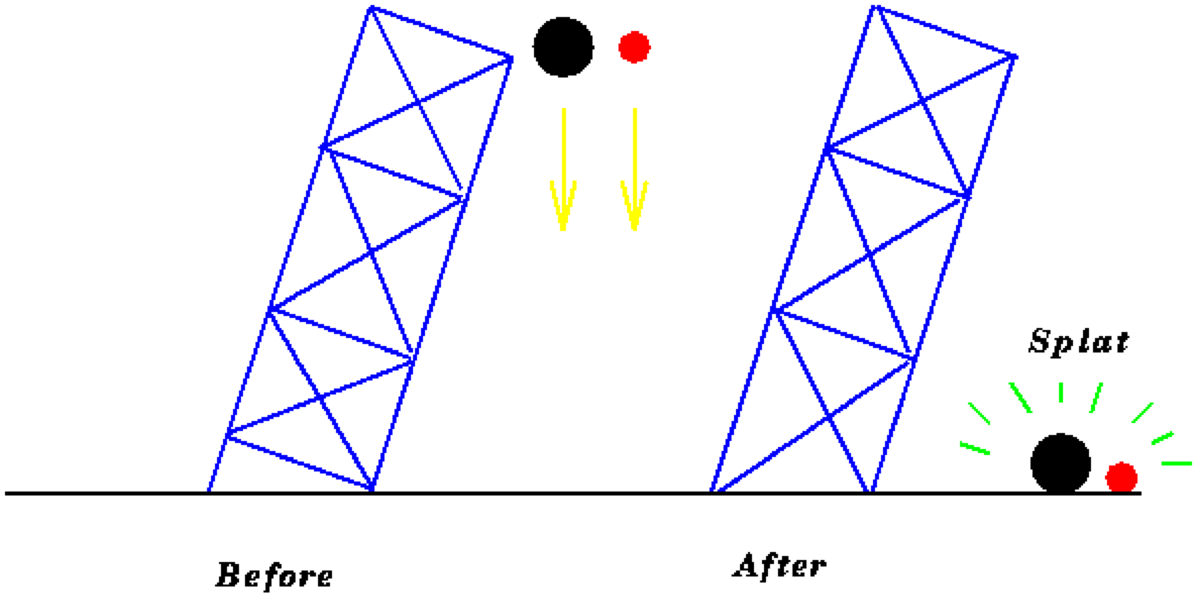
Space-time Manifold

- A **manifold** is a topological space that resembles Euclidean space near each point.
- Although a manifold resembles Euclidean space near each point, globally it may not.
- Spacetime manifold in the solar system is not like Euclidean space.
- Conclusions
  - Do not impose the Newtonian concepts in testing GR
  - Be as much close to the Newtonian concepts as possible but not closer

From Galileo to Einstein: Gravitation is not a Scalar Field!







Building Blocks of Relativity

Scalar Field ϕ $\Rightarrow$ Metric Tensor $g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$

Gravitational Force $\nabla_i \phi$ $\Rightarrow$ Affine Connection $\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\sigma}\left(g_{\sigma\alpha,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma}\right)$

Tidal Force $\nabla_i \nabla_j \phi$ $\Rightarrow$ Curvature Tensor $R^{\mu}{}_{\alpha\beta\gamma} = \Gamma^{\mu}_{\alpha\gamma,\beta} - \Gamma^{\mu}_{\alpha\beta,\gamma} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\sigma}_{\alpha\gamma} - \Gamma^{\mu}_{\gamma\sigma}\Gamma^{\sigma}_{\alpha\beta}$

Laplace's Operator $\Delta\phi$ $\Rightarrow$ Einstein Tensor $G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$

Density of Matter ρ $\Rightarrow$ Stress-Energy Tensor $T_{\alpha\beta} = \rho u_{\alpha} u_{\beta} + \pi_{\alpha\beta}$

Einstein's Field Equations and Gauge Freedom

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = \frac{8\pi G}{c^4} T^{\alpha\beta} \qquad \boxed{(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R)_{;\beta} \equiv 0} \quad \Rightarrow \quad T^{\alpha\beta}{}_{;\beta} = 0$$

Four Bianchi identities indicates the existence of the gauge freedom in the choice of the metric tensor. More specifically, any four out of ten components of the metric tensor can be chosen arbitrary. It is equivalent to the choice of a specific class of coordinate systems (CS).

$$(\sqrt{-g}\, g^{\alpha\beta})_{,\beta} = 0$$

the harmonic gauge is one of the most convenient CS.

It was recommended for use by GA of the IAU2000.

$$\gamma^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta} - \eta^{\alpha\beta}$$

$$\left(-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \Delta \right) \gamma^{\alpha\beta} = -\frac{16\pi G}{c^4} \left(T^{\alpha\beta} + \Lambda^{\alpha\beta} \right)$$

Solving Einstein's Equations

Small parameters:

$\delta =$ (size of the body)/(distance between the bodies) $\propto L/R$

$\varepsilon =$ (speed of matter)/(speed of gravity) $\propto v/c$

$\kappa =$ (gravitational radius of the body)/(size of the body) $\propto GM/c^2L$

Post - Newtonian Approximations (non - analytic expansion, elliptic equations):

$$\gamma^{\alpha\beta} = \varepsilon\gamma_1^{\alpha\beta} + \varepsilon^2\gamma_2^{\alpha\beta} + \varepsilon^3\gamma_3^{\alpha\beta} + \varepsilon^4\gamma_4^{\alpha\beta} + \varepsilon^5\gamma_5^{\alpha\beta} + \varepsilon^6\gamma_6^{\alpha\beta} + \varepsilon^7\gamma_7^{\alpha\beta} + \varepsilon^8 \ln\varepsilon\,\gamma_8^{\alpha\beta} + \ldots$$

LAGEOS, LARES

LLR

LLR, GNSS, VLBI

Post - Minkowskian Approximations (analytic expansion, hyperbolic equations):

$$\gamma^{\alpha\beta} = \kappa\gamma_1^{\alpha\beta} + \kappa^2\gamma_2^{\alpha\beta} + \kappa^3\gamma_3^{\alpha\beta} + \ldots$$

Solar system (including Earth-Moon system, space geodesy, satellite navigation) is a unique laboratory for testing GR as we have direct access and can measure all geometric and relativistic parameters from a set of independent observations and space missions.

The Residual Gauge Freedom and Coordinates

The gauge conditions simplify Einstein's equations but the residual gauge freedom remains. It allows us to perform the post-Newtonian coordinate transformations:

$$w^{\alpha} = x^{\alpha} + \xi^{\alpha}(x)$$

$$g_{\alpha\beta}(x) = G_{\mu\nu}(w)\frac{\partial w^{\mu}}{\partial x^{\alpha}}\frac{\partial w^{\nu}}{\partial x^{\beta}} = G_{\mu\nu}(w) + \xi_{\mu,\nu} + \xi_{\nu,\mu} + O(\xi^2)$$

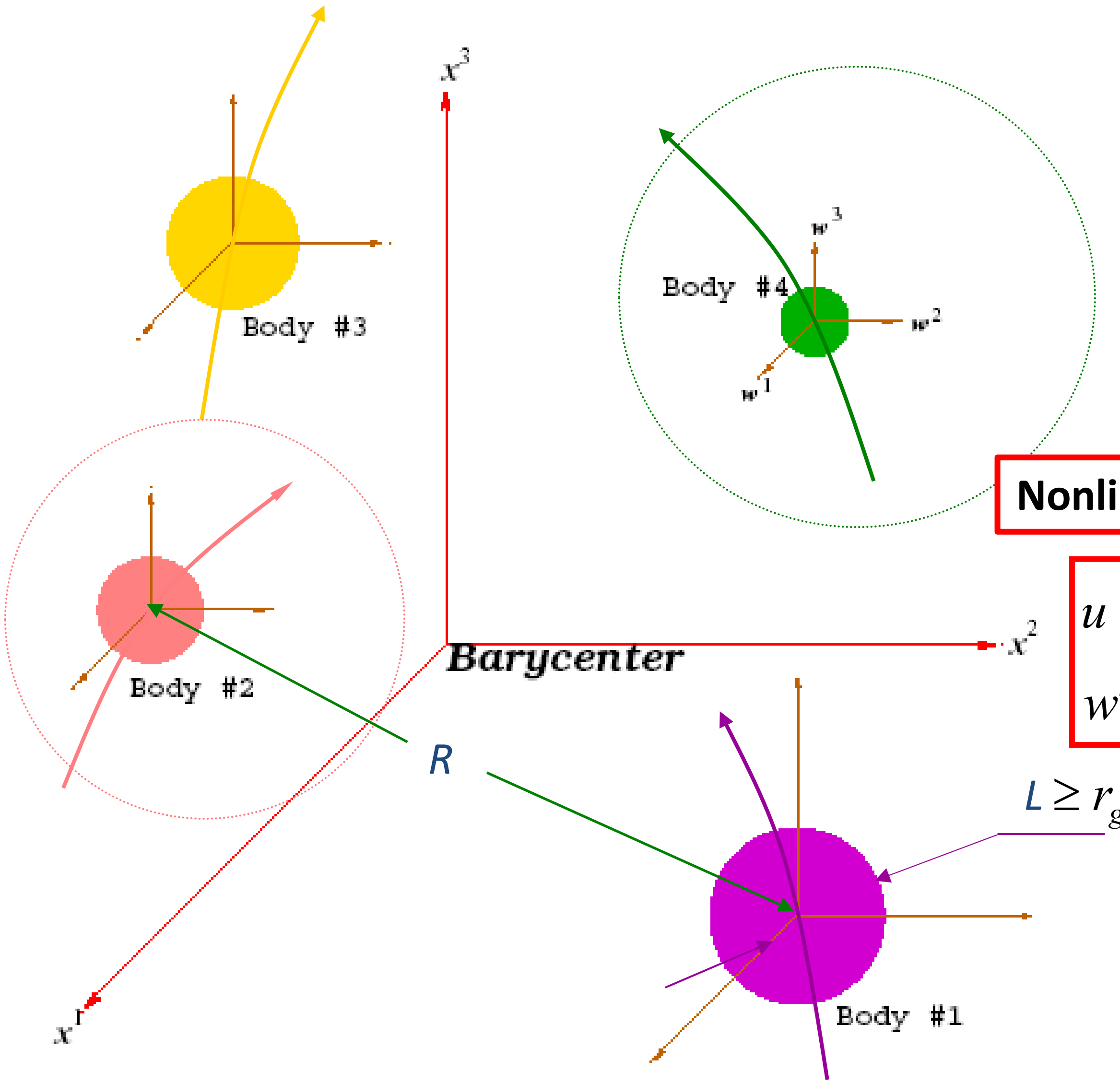
Specific choice of coordinates is determined by the boundary conditions imposed on the metric tensor components.

We - Einstein's followers - distinguish between the global and local coordinates in the sense of applicability of the Einstein principle of equivalence.

Why to introduce the local coordinates ?

- Earth-satellite/Moon system is a binary system residing on a curved space-time manifold of the solar system
- Motion of satellites are described in the most elegant way by the equation of deviation of geodesics in the presence of the (more strong) gravitational attraction of Earth.
- N-body equations of motion have enormous gauge freedom leading to the appearance of spurious, gauge-dependent forces having no direct physical meaning
- Introduction of the local coordinates is
  - to remove all gauge modes,
  - to construct and to match reference frames in the Earth-satellite/Moon system down to a millimeter precision,
  - to ensure that the observed geophysical, geodetic and orbital parameters are physically meaningful and make sense.

Global and Local Coordinates (IAU 2000 Resolutions)



$t, \vec{x}$ - barycentric coordinates

$u, \vec{w}$ - geocentric coordinates

$T, \vec{X}$ - observer' coordinates

Nonlinear coordinate transformations

$$u = u(t, \vec{x})$$
$$w^i = w^i(t, \vec{x})$$

$$T = T(u, \vec{w})$$
$$X^i = X^i(u, \vec{w})$$

The gauge freedom in SLR/LLR

$$\begin{cases} t \rightarrow t - \dfrac{1}{c^4} \displaystyle\sum_B \nu_B \dfrac{GM_B}{r_B} (\vec{r}_B \cdot \vec{v}_B) \\ \vec{x} \rightarrow \vec{x} - \dfrac{1}{c^2} \displaystyle\sum_B \lambda_B \dfrac{GM_B}{r_B} \vec{r}_B \end{cases}$$

ν_B and λ_B are gauge-fixing parameters

Here ν_B and λ_B are constant coordinate parameters which choice defines the class of a barycentric coordinate system used in SLR/LLR data processing software

$\nu_B = \lambda_B = 0$ **harmonic coordinates**

$\nu_B = 0\,;\ \ \lambda_B = 1 + \gamma$ **Painlevé coordinates**

EIH equations of motion in the barycentric coordinates

Kopeikin, PRL, 98, Issue 22, id. 229001 (2007); Kopeikin & Yi, CMDA, 108, 245-263 (2010)

$$a_B^i = \sum_{C\neq B}\left[E_{BC}^i + \frac{2+2\gamma-2\lambda_C}{c}\left(v_B \times H_{BC}\right)^i - \frac{1+2\gamma-2\lambda_C}{c}\left(v_C \times H_{BC}\right)^i \right]$$

$$E_{BC}^i = -\frac{GM_C}{R_{BC}^3} R_{BC}^i \left(1 + \frac{\mathcal{E}_{BC}}{c^2}\right)$$

Gravito-electric force

$$H_{BC}^i = -\frac{1}{c}\left(V_{BC} \times E_{BC}\right)^i$$

Gravito-magnetic (orbital motion-induced) force

$$\begin{aligned}\mathcal{E}_{BC} = &\left(-1-2\gamma+3\lambda_C\right)v_B^2 + \left(1+2\gamma-6\lambda_C\right)\left(v_B \cdot v_C\right) - \left(\gamma-3\lambda_C\right)v_C^2 - \frac{3}{2}\left(N_{BC}\cdot v_C\right)^2 \\ &-3\lambda_C\left(N_{BC}\cdot V_{BC}\right)^2 - \left(1+2\gamma+2\beta-2\lambda_B\right)\frac{GM_B}{R_{BC}} - 2\left(\gamma+\beta-\lambda_C\right)\frac{GM_C}{R_{BC}} - \\ &-\sum_{D\neq B,C} GM_D R_{BC}^3\left[\frac{1-2\beta+2\lambda_D}{R_{CD}R_{BC}^3} - \frac{2\left(\gamma+\beta\right)-\lambda_D}{R_{BD}R_{BC}^3} - \frac{2\left(\gamma+1\right)}{R_{BC}R_{CD}^3} - \frac{\lambda_C}{R_{BC}R_{BD}^3} + \frac{\lambda_C}{R_{CD}R_{BD}^3} + \frac{3+4\gamma-2\lambda_D}{2R_{BD}R_{CD}^3}\right] \\ &+\sum_{D\neq B,C} GM_D\left(R_{BC}\cdot R_{BD}\right)\left[\frac{1+2\lambda_C}{2R_{CD}^3} - \frac{\lambda_C}{R_{BD}^3} + \frac{3\lambda_D}{R_{BD}R_{BC}^2} - \frac{3\lambda_D}{R_{CD}R_{BC}^2}\right]\end{aligned}$$

Observables are gauge invariant

$$t_2 - t_1 = \frac{R_{12}}{c} + \overset{(1+\gamma)}{\overset{\downarrow}{2}} \sum_B \frac{GM_B}{c^3} \ln\left[\frac{R_{1B} + R_{2B} + R_{12}}{R_{1B} + R_{2B} - R_{12}}\right] + \sum_B \nu_B \frac{GM_B}{c^4}\left[\frac{\boldsymbol{R}_{2B} \cdot \mathbf{v}_B(t_2)}{R_{2B}} - \frac{\boldsymbol{R}_{1B} \cdot \mathbf{v}_B(t_1)}{R_{1B}}\right]$$

$$+ \sum_B \lambda_B \frac{GM_B}{c^3} \frac{(R_{1B} - R_{2B})^2 - R_{12}^2}{2R_{1B} R_{2B} R_{12}} (R_{1B} + R_{2B})$$

t_1 - time of photon's emission at point $\overrightarrow{x_1}$

t_2 - time of photon's reception at point $\overrightarrow{x_2}$

The gauge-fixing parameters ν_B and λ_B enters both the N-body equations of motion and the equations of light propagation. All together it makes the procedure of fitting the measured parameters to SLR/LLR data gauge-invariant.

Toward a better SLR/LLR relativistic model

LLR test of General Relativity is far from being completed as the currently employed data processing algorithm does not distinguish between the spurious coordinate-dependent forces and the true (curvature related) gravitational forces.

To separate the spurious forces, being dependent on the choice of coordinates, the relativistic theory of local frames must be employed (see the textbook by Kopeikin, Efroimsky, Kaplan "*Relativistic Celestial Mechanics of the Solar System*" Wiley, 2011)

There are other problems with the interpretation of the measurement of SEP and/or Gdot as we need a much more consistent theory of these violations (see "*Frontiers in Relativistic Celestial Mechanics*" ed. S. Kopeikin, De Gruyter, 2014)

Radial (synodic) relativistic effects in the orbital motion of satellite/Moon

Schwarschild	$\boxed{\dfrac{GM_\oplus}{c^2}}$	~ 1 cm
Lense-Thirring	$\boxed{\dfrac{\omega_\oplus R_\oplus}{c}\dfrac{v}{c}R_\oplus}$	$\sim 2.1/0.3\ mm$
PN Quadrupole	$\boxed{\dfrac{GM_\oplus}{c^2}\left(\dfrac{R_\oplus}{r}\right)^2 J_{2\oplus}}$	$\sim 10^{-2}/\ 10^{-4}\ mm$
Gauge-dependent terms	$\boxed{\dfrac{v}{c}\dfrac{v_\oplus}{c}r+\ldots}$	~ from a few meters down to a few cm
Tidal gravito-magnetic	$\boxed{\left(\dfrac{n_S}{n_\oplus}\right)^2\dfrac{v}{c}\dfrac{v_\oplus}{c}r}$	$\sim 0.1\ /\ 1\ mm$
Tidal gravito-electric	$\boxed{\left(\dfrac{n_S}{n_\oplus}\right)^2\left(\dfrac{v_\oplus}{c}\right)^2 r}$	~ a few cm
Non-linearity of gravity	$\boxed{\left(\dfrac{n_S}{n_\oplus}\right)^2\dfrac{GM_\oplus}{c^2}}$	~ 0.1 mm

Relativistic Geoid

Kopeikin S., Manuscripta Geodaetica, vol. 16, 301 - 312 (1991)   (theory in progress)

Definition 1: The relativistic *a*-geoid represents a two-dimensional surface at any point of which the direction of plumb line measured by a static observer is orthogonal to the tangent plane of the geoid's surface.

Definition 2: The relativistic *p*-geoid represents a two-dimensional level surface of a constant pressure of the rigidly rotating perfect fluid.

Definition 3: The relativistic *u*-geoid represents a two-dimensional surface at any point of which the rate of the proper time, τ, of an ideal clock carried out by static observers with fixed geodetic coordinates r, θ, ϕ, is constant.

THANK YOU!