New Dynamical Relativistic Modeling of the Moon Orbital and Rotational Motion developed by POLAC (Paris Observatory Lunar Analysis Center)

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Summary

1. Software Features
   - Numerical Integration
   - Fitting the Numerical Solution
2. Physical Dynamical Model
   - The Main Problem
   - List of effects
   - Comparison with INPOPXX/DEXXX
3. Some Applications
   - Improvement of the Semi-Analytical ELP Theory
   - Test of Gravity in Weak Field

Summary

Numerical Integration

ODEX Integrator

E. Hairer et al., Solving Ordinary Differential Equation I, 1993

Optimized for quadruple precision (10^{-32}).

Dimensionless Equations of Motion

We compute *forces* and *momentums* in the ICRS (Solar System Barycenter) by substituting variables :

$$\boldsymbol{r} \longrightarrow r_c\, \boldsymbol{\chi} \quad \text{and} \quad t \longrightarrow t_c\, \tau, \quad \text{then,} \quad \mathrm{d}^2\boldsymbol{r} = r_c\, \mathrm{d}^2\boldsymbol{\chi} \quad \text{and} \quad \frac{1}{\mathrm{d}t^2} = \frac{1}{t_c^2} \times \frac{1}{\mathrm{d}\tau^2}.$$

The Dimensionless 2 Bodies Problem

$$\frac{\mathrm{d}^2\boldsymbol{\chi}}{\mathrm{d}\tau^2} = -\frac{\mu t_c^2}{r_c^3}\frac{\boldsymbol{\chi}}{\|\boldsymbol{\chi}\|^3}, \quad \text{we put :} \quad \frac{\mu t_c^2}{r_c^3} = 1 \quad \text{with} \quad \mu = G(M_T + M_L)$$

Identifying to Kepler 3rd law : $\mu = n^2 a^3$, we find $t_c = 1/n$ and $r_c = a$.

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Identifying to Kepler 3rd law : $\mu = n^2 a^3$, we find $t_c = 1/n$ and $r_c = a$.

Fitting the Numerical Solution

Observables

- The dynamical model is fitted to an ephemeride (INPOPXX, ELPXX, DEXXX). Several possibilities for the minimisation of observables :
  - $||\boldsymbol{EM}||^{\text{eph}} - ||\boldsymbol{EM}||^{\text{int}}$
  - $||\mathbf{0} - \boldsymbol{M}^{\text{eph}}\boldsymbol{M}^{\text{int}}||$
  - ${}^{T}(\phi, \theta, \psi)^{\text{eph}} - {}^{T}(\phi, \theta, \psi)^{\text{int}}$ or $\cos\varphi^{\text{eph}} - \cos\varphi^{\text{int}}$

  with $\cos\varphi = \cos\phi\cos\psi + \sin\phi\sin\psi\cos(\pi - \theta)$.
- Fit of the solution to LLR datas thanks CAROLL.

Least-Squares Method

$$g(\boldsymbol{x}) = \frac{1}{2}||f(\boldsymbol{x}) - \boldsymbol{y}||^2 \in \mathbb{R}^+ \Longrightarrow g'(\boldsymbol{x}) = 0$$

where e.g. $\boldsymbol{y} = ||\boldsymbol{EM}||_t^{\text{eph}}$ and $f(\boldsymbol{x}) = ||\boldsymbol{EM}||_t^{\text{int}}$ with $t = 0, \cdots, t_{\text{fin}}$. If $\boldsymbol{x_0}$ is a closed solution of $\boldsymbol{x}$, we have $\boldsymbol{\delta x} = \boldsymbol{x} - \boldsymbol{x_0}$ and approximating g by it Taylor expansion give

$$\boldsymbol{\delta x} = \{{}^{T}f'(\boldsymbol{x_0}) \cdot f'(\boldsymbol{x_0})\}^{-1} \cdot \{{}^{T}f'(\boldsymbol{x_0}) \cdot [\boldsymbol{y} - f(\boldsymbol{x_0})]\}$$

where $f'(\boldsymbol{x_0})$ is the partial derivative matrix.

Computation of Matrix $f'(\boldsymbol{x})$

Initial Solution Vector

$\boldsymbol{x} = {}^T(p^1, \cdots, p^m, \chi^1, \cdots, \chi^3, \dot{\chi}^1, \cdots, \dot{\chi}^3)$. $\boldsymbol{p}$ physical parameters vector, $\boldsymbol{\chi}$ dimensionless position vector and $\dot{\boldsymbol{\chi}}$ dimensionless velocity vector.

Numerically

$$\frac{\partial f(\boldsymbol{x})}{\partial x^k} = \frac{f(x^1, \cdots, x^k + \epsilon, \cdots, x^{m+6}) - f(x^1, \cdots, x^k - \epsilon, \cdots, x^{m+6})}{2\epsilon} + o(\epsilon^2)$$

From the Variation's Equation

$l = 1, \cdots, m+6$ et $i = 1, \cdots, 3$ we can integrate the $6 \times (m+6)$ following equations :

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \chi^i}{\partial x^l}\right) = \dot{\mathrm{X}}^i_l \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\mathrm{X}}^i_l\right) = \displaystyle\sum_{k=1}^{3} \frac{\partial \mathcal{A}^i}{\partial \chi^k}\frac{\partial \chi^k}{\partial x^l} + \sum_{k=1}^{3} \frac{\partial \mathcal{A}^i}{\partial \dot{\chi}^k}\frac{\partial \dot{\chi}^k}{\partial x^l} + \sum_{k=1}^{m} \frac{\partial \mathcal{A}^i}{\partial p^k}\frac{\partial p^k}{\partial x^l} \end{cases}$$

where $\boldsymbol{\mathcal{A}} = \ddot{\boldsymbol{\chi}}(\boldsymbol{x}) = \ddot{\boldsymbol{\chi}}(\boldsymbol{p}, \boldsymbol{\chi}, \dot{\boldsymbol{\chi}})$ is the dimensionless Earth-Moon acceleration vector.

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Modeling

Main Problem

Earth, Moon and Sun are considered as ponctual masses. Sun describes a keplerian ellipse, the ecliptic, around the Earth-Moon barycenter.

Restricted Planetary Perturbations

Situation is the same as before, and planetary perturbations are considered. Planets and Pluto describe keplerian ellipses around Sun.

Forced Planetary Perturbations

Only the Earth-Moon vector is integrated. Planetary perturbations are directly taken from an existing planetary ephemeride at each time step.

N Bodies Problem

We compute the Newtonian interactions between all planets, Moon, Sun and Pluto.

Relativistic Interactions

Masse/Energy of the System

$$\mathcal{M} = \sum_{\mathrm{A}} \mu^*_{\mathrm{A}} \quad \text{with} \quad \mu^*_{\mathrm{A}} = \mu_{\mathrm{A}} \left\{ 1 + \frac{\dot{r}^2_{\mathrm{A}}}{2c^2} - \frac{1}{2c^2} \sum_{B \neq A} \frac{\mu_{\mathrm{B}}}{r_{\mathrm{AB}}} + O(c^{-4}) \right\}$$

SSB secular drift of 0.5 mm/century

Solar System Barycenter

1. Position and velocity of *SSB* are taken from a given planetary ephemeride at each time step.
2. We impose *SSB* to be fixed at the frame's origin (ICRS) at each time step.
3. We impose *SSB* to be fixed at the frame's origin (ICRS) at beginning of the integration.

Description

We compute the relativistic effects of all bodies through a Parametrized Post-Newtonian modeling ($\beta = 1$, $\gamma = 1$), considering General Relativity as a perturbation to Newtonian gravity. Each body is considered as generating a locally spherically symmetric gravitationnal field.

Relativistic Interactions

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$$\mathcal{M} = \sum_{\mathsf{A}} \mu_{\mathsf{A}}^* \quad \text{with} \quad \mu_{\mathsf{A}}^* = \mu_{\mathsf{A}} \left\{ 1 + \frac{\dot{r}_{\mathsf{A}}^2}{2c^2} - \frac{1}{2c^2} \sum_{B \neq A} \frac{\mu_{\mathsf{B}}}{r_{\mathsf{AB}}} + O(c^{-4}) \right\}$$

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Earth Figure

Spherical Harmonic

The Earth is no more considered as a ponctual mass but as an axisymmetric body. We consider only zonal harmonics up degree 4 (J_2, J_3, J_4).

Earth Nutations and Precession Motion

- Earth precession motion is forced with P03 solution (*Capitaine et al., 2003*).
- Earth nutations are forced with *SOFA*'s routines (*Standards Of Fundamental Astronomy*) *http ://www.iausofa.org/*

Tides

Love's Hypothesis

Each harmonic corresponding to the variations of the potential of a body (Earth e.g.) at its surface $(r' = R, \varphi', \lambda')$ is proportional to the same harmonic of the potential of a perturber (Moon or Sun e.g.).

$$\Delta\psi(R, \varphi', \lambda') = \sum_{n=2}^{+\infty} \sum_{m=0}^{n} \Delta\psi_{nm}(R, \varphi', \lambda') = \sum_{n=2}^{+\infty} \sum_{m=0}^{n} k_{nm} V_{nm}(R, \varphi', \lambda')$$

Dissipation

The deformation of a body is not immediate, the response is related to its internal structure and produces dissipation. It leads to a time delay (τ_{nm}) between the bulge and the perturber direction. So, the deformation due to a perturber at time t is due to its position at time $t - \tau_{nm}$.

Tides on Earth

Tides DE200

The Moon is the only tide raising body on Earth and the only interacting body. The Earth is considered elastic with only one k and one τ. During the time τ, the orbital motion of the Moon is neglected and we suppose that the phase lag is only due to Earth spin ($\dot{\psi} \sim cst$). So, phase angle between the tidal bulge and the Earth-Moon vector ($\delta = \tau\dot{\psi}$) is constant.

Tides DE405

The Moon and the Sun are the tide raising bodies on Earth. The Moon is the only interacting body with the tidal bulge. We consider only the modification of degree 2 of Earth potential. This modification is taken into account through the variation of the potential Stokes coefficients considering anelastic Earth.

$$\begin{cases} C_{2m}(t) = C^{\mathrm{R}}_{2m} + \Delta C^{\mathrm{T}}_{2m}\left(k_{2m},\ \boldsymbol{r}_{\mathrm{g}}(t-\tau_{2m})\right) \\ S_{2m}(t) = S^{\mathrm{R}}_{2m} + \Delta S^{\mathrm{T}}_{2m}\left(k_{2m},\ \boldsymbol{r}_{\mathrm{g}}(t-\tau_{2m})\right) \end{cases} \quad \text{with} \quad m = 0, 1, 2$$

For Earth, the delayed position is evaluated at second order in τ_{2m}

$$\boldsymbol{r}_{\mathrm{g}}(t-\tau_{2m}) = \boldsymbol{r}_{\mathrm{g}}(t) - \tau_{2m}\,\dot{\boldsymbol{r}}_{\mathrm{g}}(t) + \frac{\tau_{2m}^2}{2}\,\ddot{\boldsymbol{r}}_{\mathrm{g}}(t) + o(\tau_{2m}^2)$$

Moon Potential

Spherical Harmonic

The Moon is also considered as an extended body. We consider all harmonics up degree 4 (C_{nm} and S_{nm} with $n = 2 \rightarrow 4$ and $m = 0 \rightarrow n$).

Non-Rigid Moon

The Moon is considered as an elastic body. We consider deformations due to tidal effects (Earth and Sun are the tide raising bodies) and spin. $\boldsymbol{\omega}$ represents the spin velocity vector of the Moon expressed in the frame rotating with Moon.

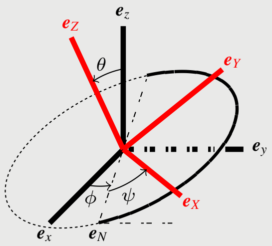
$$\begin{cases} C_{2m}(t) = C^{\mathrm{R}}_{2m} + \Delta C^{\mathrm{T}}_{2m}\,(k_{\mathrm{M}},\ \boldsymbol{r}_{\mathrm{g}}(t-\tau_{\mathrm{M}})) + \Delta C^{\mathrm{S}}_{2m}\,(k_{\mathrm{M}},\ \boldsymbol{\omega}(t-\tau_{\mathrm{M}})) \\ S_{2m}(t) = S^{\mathrm{R}}_{2m} + \Delta S^{\mathrm{T}}_{2m}\,(k_{\mathrm{M}},\ \boldsymbol{r}_{\mathrm{g}}(t-\tau_{\mathrm{M}})) + \Delta S^{\mathrm{S}}_{2m}\,(k_{\mathrm{M}},\ \boldsymbol{\omega}(t-\tau_{\mathrm{M}})) \end{cases} \quad \text{with} \quad m = 0, 1, 2$$

For Moon, $\boldsymbol{r}_{\mathrm{g}}(t-\tau_{\mathrm{M}})$ and $\boldsymbol{\omega}(t-\tau_{\mathrm{M}})$ are computed by RK4 integration between t and $t - \tau_{\mathrm{M}}$.

Moon Librations

Euler Angles

Moon's orientation ($\mathcal{R}_{\mathrm{M}}$) can be identified in the ICRS ($\mathcal{R}_{\mathrm{ICRS}}$) thanks to Euler angles.



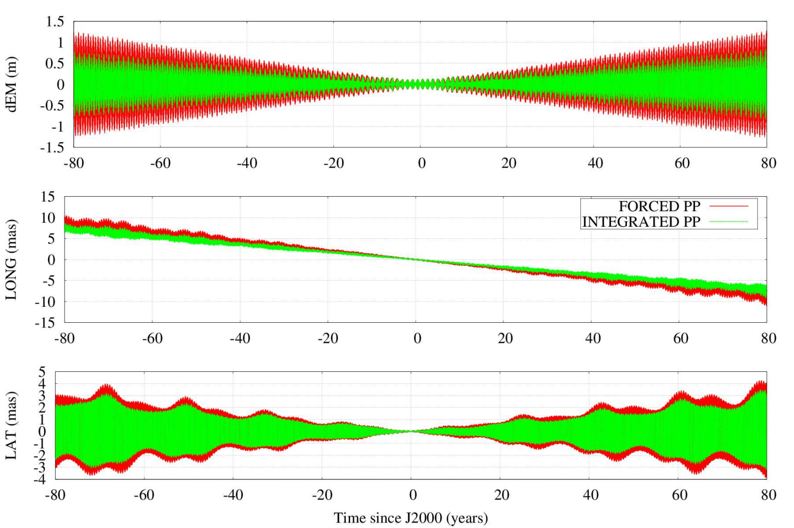
Euler Equations of Motion

We integrate ${}^{\mathrm{T}}(\ddot{\phi}, \ddot{\theta}, \ddot{\psi}) = f(\dot{\boldsymbol{\omega}}, \phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi})$ by computing $\dot{\boldsymbol{\omega}}$.

$$\dot{\boldsymbol{\omega}} = \boldsymbol{\mathcal{I}}^{-1}.\left\{\sum \boldsymbol{\mathcal{N}}_{\mathrm{ext}} - \dot{\boldsymbol{\mathcal{I}}}.\boldsymbol{\omega} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{I}}.\boldsymbol{\omega}\right\}$$

Moon inertia matrix is time varying $\boldsymbol{\mathcal{I}}(t)$ and depends on $C_{2m}(t)$ and $S_{2m}(t)$.

DE405-Integration without fit



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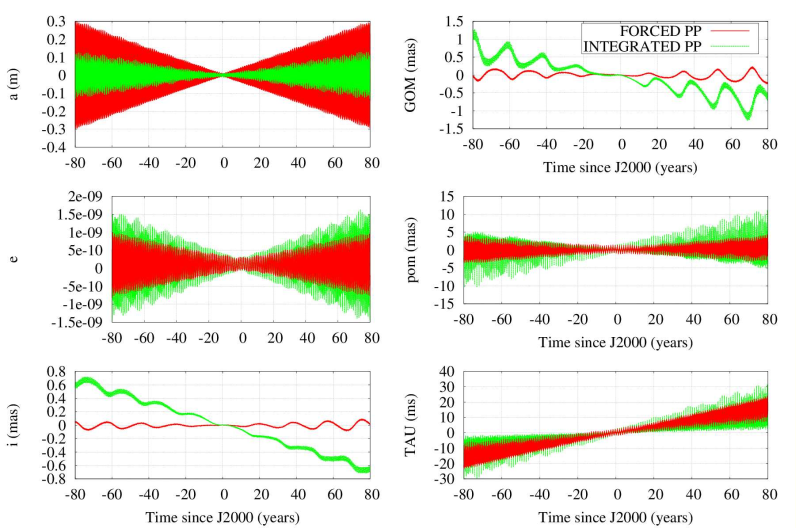

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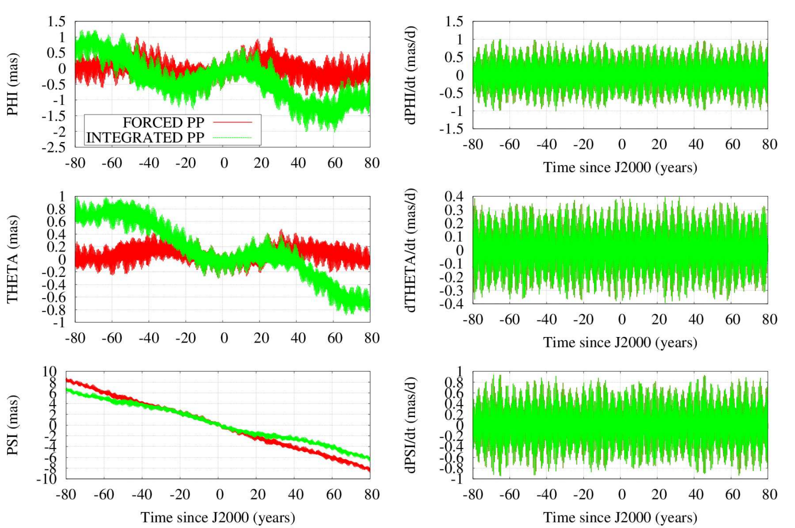


Summary

ELP

Historic

Developped since the 70's by J. Chapront, M. Chapront-Touzé and G. Francou, it is a semi-analytical solution to the dynamical equations governing the Moon gravity center motion. The last solution (ELP-MPP02, 2003) takes into account all effects with theoretical signal larger than one millimeter over the Earth-Moon vector. By decreasing importance,

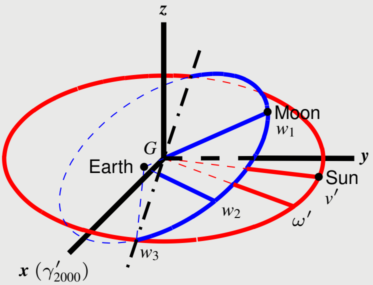
- The Main Problem ($10^8 - 10^9$ cm)
- Planetary Perturbations ($10^4 - 10^5$ cm)
- Earth Figure ($10^4 - 10^5$ cm)
- General Relativity ($10^2 - 10^3$ cm)
- Tide on Earth ($10^2 - 10^3$ cm)
- Effects due to the Moon Orientation ($10^1 - 10^2$ cm).

ELP-MPP02 is the ephemeride currently used by POLAC for the reduction and the analyse of LLR datas.

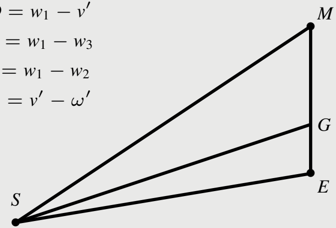
Bidart, P., 2001, A&A, 366, 351.
Chapront-Touzé, M., Chapront, J., 1988, A&A, 190, 342.
Chapront-Touzé, M., Chapront, J., 1997, Celest.Mech., 66, 31.
Chapront, J., Chapront-Touzé, M., Francou G., 2002, A&A, 387, 700.

ELP Solution

Reference Plane (Centered on Earth, parallel to ecliptic)



$$D = w_1 - v'$$
$$F = w_1 - w_3$$
$$l = w_1 - w_2$$
$$l' = v' - \omega'$$



Solution to the Main Problem

$$\sum_{i_1,\cdots,i_4} [A^{i_1,\cdots,i_4} + \sum_j B_j^{i_1,\cdots,i_4} \delta x_j^0]. \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (i_1\bar{D} + i_2\bar{F} + i_3\bar{l} + i_4 l')$$

$A^{i_1,\cdots,i_4}$ and $B_j^{i_1,\cdots,i_4}$ are numerical coefficients and δx_j^0 are literal variations of the constants used for the construction of the solution.

Main Limitations

Comparison with INPOP08

In his pure semi-analytical form it does not reach a millimetric precision. The RMS over +100/-100 years after fitting to INPOP08 gives

$$\Delta V = 1.56 \text{ mas} \quad \Delta U = 0.56 \text{ mas} \quad \Delta r = 71 \text{ cm}$$

Explanations

- Slow convergence of Poisson series (e.g. planetary effects).
- Truncation of series for their computation.
- Implicit physical parameters impossible to fit.
- Implicit physical parameters impossible to update.

Solutions with Numerical Integration

- Comparing Numerical to semi-analytical, effect by effect, can provide the true precision of a series.
- Replacing a too slow convergent series by its numerical counterpart (e.g. planetary effects).

Testing Metric Theories of Gravity

Principle

Be able to simulate observables in different theories of gravity. Compute equations of motion from a given metric theory.

Weak Field

$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ with $|h_{\alpha\beta}| \ll 1$ and $\eta_{\alpha\beta}$ Minkowski metric tensor : $\eta_{00} = 1$; $\eta_{0i} = 0$; $\eta_{ij} = -\delta_{ij}$

Geodesics Equation

Expressed in function of the coordinate time $x^0 = ct$:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}x^{0\,2}} = -\Gamma^i_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}x^0} \frac{\mathrm{d}x^\beta}{\mathrm{d}x^0} + \Gamma^0_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}x^0} \frac{\mathrm{d}x^\beta}{\mathrm{d}x^0} \frac{\mathrm{d}x^i}{\mathrm{d}x^0}$$

Testing Metric Theories of Gravity

Equations of motion

$$\ddot{x}^i = -\frac{c^2}{2}\partial_i h_{00} - \frac{c^2}{2} h_{ik}\partial_k h_{00} + \frac{c^2}{2}\partial_0 h_{0i} + \frac{c}{2}\partial_0 h_{00}\dot{x}^i + c\partial_0 h_{ik}\dot{x}^k + c(\partial_k h_{0i} - \partial_i h_{0k})\dot{x}^k + \partial_k h_{00}\dot{x}^k\dot{x}^i + (\partial_m h_{ik} - \frac{1}{2}\partial_i h_{km})\dot{x}^k\dot{x}^m + O(c^{-4}).$$

with $\dot{x}^i = \mathrm{d}x^i/\mathrm{d}t$. Knowing $h_{\alpha\beta}$ et $h_{\alpha\beta,\gamma}$ we can deduce the equations of motion in any metric theory of gravity which can be expressed under the form $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$. Actually, $h_{\alpha\beta} = h'_{\alpha\beta} + \delta h_{\alpha\beta}$ where $h'_{\alpha\beta}$ corresponds to General Relativity and $\delta h_{\alpha\beta}$ to the alternative theory.

A. Hees et al., Radioscience simulations in general relativity and in alternative theories of gravity, 2012, Classical and Quantum Gravity

Thank you for your attention !